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Anomaly of Discrete Symmetries and Gauge Coupling Unification

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Abstract

Anomaly of a discrete symmetry is defined as the Jacobian of the path-integral measure. Assuming that anomaly at low energy is cancelled by the Green-Schwarz (GS) mechanism at a fundamental scale, we investigate possible Kac-Moody levels for anomalous discrete family symmetries. As the first example we consider discrete abelian Baryon number and Lepton number symmetries in the minimal supersymmetric standard model with see-saw mechanism, and find that the ordinary unification of gauge couplings is not consistent with the GS conditions, indicating a possible existence of further Higgs doublets. We consider various recently proposed supersymmetric models with a non-abelian discrete family symmetry. In a supersymmetric example with Q_6 family symmetry, the GS conditions are such that the gauge coupling unification appears close to the Planck scale.

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1 Introduction

Though the standard model (SM) is very successful, there are many unsatisfactory features. One of them is the redundancy of the free parameters in the Yukawa sector. There exist (infinitely) many physically equivalent Yukawa matrices that can produce the same physical quantities such as the fermion masses and mixings. Since there is no principle to fix the Yukawa structure in the SM, one can extend the SM to impose a family symmetry to reduce the redundancy of the parameters. Recently, non-abelian discrete symmetries have been used to extend the SM [1, 2, 3, 4, 5, 6, 7].

Another unsatisfactory thing is the fine-tuning problem of the Higgs mass, which can be softened by supersymmetry (SUSY). Moreover, the successful gauge coupling unification in the minimal supersymmetric standard model (MSSM) seems to support the existence of low-energy SUSY. However, the situation is not that simple, because SUSY is broken at low energy. It is widely assumed that to maintain the nice renormalization property of a supersymmetric theory SUSY must be broken softly. It is known that there are more than one hundred soft breaking terms and they, unless they are fine tuned, cause large flavor changing neutral currents (FCNC) and CP violations. Fortunately, there is a good news that this SUSY flavor problem may be softened by a non-abelian family symmetry [8].

These facts suggest that a family symmetry could cure certain pathologies of the SM and MSSM. This leads us to assume that a family symmetry consistent at low energy is remnant of a symmetry of a more fundamental theory. If this is the case, the symmetry should be anomaly free at least at a fundamental scale. This is why we are going to investigate anomalies of discrete family symmetries of recently proposed models. In this paper we assume that anomaly of a discrete symmetry at low-energy is canceled by the Green-Schwarz(GS) mechanism [9, 10, 11] at a more fundamental scale. In this scenario the Kac-Moody k_i levels play an important role, and if the Kac-Moody levels assume nontrivial values, the GS cancellation conditions of anomalies modify the ordinary unification of gauge couplings. Note that it is impossible to construct a realistic, renormalizable model with a low-energy non-abelian discrete family symmetry in the case of the minimal content of the $SU(2)_L$ Higgs fields. However, an extension of the Higgs sector may spoil the successful gauge coupling unification of the MSSM. Therefore, a possible change of the ordinary unification condition because of nontrivial Kac-Moody k_i levels is fit to the assumption on the existence of a low-energy non-abelian discrete family symmetry.

Anomaly of discrete symmetries are already discussed in Ref. [12, 13, 14, 15]. In these papers, it is assumed that all discrete symmetries in low energy should be gauged at high energy as it is also assumed in [16]. In other words, they assume that to survive quantum gravity effects such as wormholes [17], all low energy discrete symmetries must be generated from a spontaneous break down of continuous gauge symmetries.

In sect. 2 we calculate, using Fujikawa's method [18], the Jacobian of the path-integral measure of an anomalous abelian discrete symmetry. We do so in this section to recall and to demonstrate that the Jacobian can be calculated for a finite discrete transformation parameter. In sect. 3 we use the result of sect. 2 to calculate the Jacobian for a non-abelian discrete transformation. Anomaly cancellation is discussed in sect. 5. First we recall the case of an anomalous $U(1)$, and then we extend the cancellation mechanism to the case of discrete symmetries. In contrast to the treatment of [14] we do not assume that the discrete symmetry in question is not remnant of a spontaneously broken continuous symmetry. As the first example we consider discrete abelian Baryon number and Lepton number symmetries in the minimal supersymmetric standard model with see-saw mechanism in sect. 6. We find that the GS cancellation conditions can be satisfied if $k_3/k_2 = \text{even/odd}$. This implies that the ordinary unification of gauge couplings is not consistent with the GS conditions, indicating a possible existence of further Higgs doublets. We investigate the unification of gauge couplings for $k_3/k_2 = 2$ and find that the unification scale appears close to the Planck scale for three pairs of $SU(2)_L$ doublet Higgs supermultiplets. The recent models with D_7, A_4 and Q_6 symmetries are treated in sect. 7, and sect. 8 is devoted for summary.

2 Anomaly of Abelian Discrete Symmetries

Anomaly is a violation of a symmetry at the quantum level. In the case of a continuous symmetry, anomaly means non-conservation of the corresponding Noether current. For discrete symmetries, however, we can not define anomaly in this way because there are no corresponding Noether currents. But Fujikawa's method [18], which is based on the calculation of the Jacobian of the path-integral measure, can be used to define anomaly of discrete symmetries. As we will see below, the calculation is basically the same as the conventional method [18].

Let us start by considering a Yang-Mills theory with massless fermions ψ in the Euclidean space time, which can be described by the following path-integral with the Lagrangian:

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathbf{exp}\Big[\int d^4x \mathcal{L}\Big], \tag{1}$$

$$\mathcal{L} = i\bar{\psi} \not{D}\psi - \frac{1}{2g^2}\mathbf{Tr} \ F^{\mu\nu}F_{\mu\nu}, \tag{2}$$

$$D_\mu = \partial_\mu - iA_\mu, \tag{3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \ A_\nu], \tag{4}$$

$$A_\mu \equiv gT^a A^a_\mu \ , \quad \mathbf{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}, \tag{5}$$

where we have dropped the path-integral measure of the gauge boson A_μ, because it does not contribute to anomaly. Then we make a chiral phase rotation

$$\psi \to \psi' = \mathbf{e}^{i\alpha\gamma_5}\psi, \tag{6}$$

where α is a finite discrete parameter. Under this transformation, the Lagrangian is be invariant. Next we consider the transformation property of the path-integral measure. To this end, we follow [18] and define the eigenstates $\varphi_n(x)$ of $\not{D}$, i.e.,

$$\not{D}\varphi_n(x) = \lambda_n\varphi_n(x), \qquad \varphi^\dagger_n \ \not{D} = \lambda_n\varphi^\dagger_n(x), \tag{7}$$

with the relations

$$\psi_i(x) = \sum_n a_n \varphi_{n,i}(x), \qquad \bar{\psi}_i(x) = \sum_n \bar{b}_n \varphi^\dagger_{n,i}(x), \tag{8}$$

$$\int d^4x \; \varphi^\dagger_{n,i}(x)\varphi_{m,i}(x) = \delta_{nm}, \tag{9}$$

$$\sum_n^\infty \varphi_{n,i}(x)\varphi^\dagger_{n,j}(y) = \delta_{ij}\delta^4(x-y), \tag{10}$$

where i, j are spinor indices. The Jacobian of the path-integral measure for the above transformation, which is defined as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \frac{1}{J}\mathcal{D}\bar{\psi}\mathcal{D}\psi, \tag{11}$$

can be written as

$$J^{-1} = \left\{ \mathbf{det} \int d^4x \; \varphi^\dagger_{n,i}(x) \left[\mathbf{e}^{i\alpha\gamma_5}\right]_{ij} \varphi_{m,j}(x) \right\}^{-2} \equiv [\mathbf{det}\; C_{nm}]^{-2}, \tag{12}$$

where

$$C_{nm} \;\; = \;\; \int d^4x \; \varphi^\dagger_{n,i}(x) \left[\mathbf{e}^{i\alpha\gamma_5}\right]_{ij} \varphi_{m,j}(x). \tag{13}$$

C_{nm} is defined as the expansion

$$\begin{aligned} C_{nm} \;\; = \;\; & \delta_{nm} + \int d^4x \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]_{ij} \varphi_{m,j}(x) \\ & + \int d^4x \; \varphi^\dagger_{n,i}(x) \frac{1}{2!} \left[i\alpha\gamma_5\right]^2_{ij} \varphi_{m,j}(x) \\ & + \int d^4x \; \varphi^\dagger_{n,i}(x) \frac{1}{3!} \left[i\alpha\gamma_5\right]^3_{ij} \varphi_{m,j}(x) + \cdots. \end{aligned} \tag{14}$$

To proceed we first derive the following identity by using the completeness relation Eq. (10):

$$\begin{aligned} & \int d^4x \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]^2_{ij} \varphi_{m,j}(x) \\ & \quad = \int d^4x \int d^4y \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]_{ij} \delta_{jk}\delta^4(x-y) \left[i\alpha\gamma_5\right]_{kl} \varphi_{m,l}(y) \\ & \quad = \int d^4x \int d^4y \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]_{ij} \varphi_{p,j}(x)\varphi^\dagger_{p,k}(y) \left[i\alpha\gamma_5\right]_{kl} \varphi_{m,l}(y) \\ & \quad = \tilde{C}_{np}\tilde{C}_{pm} = \tilde{C}^2_{nm}, \end{aligned} \tag{15}$$

where

$$\tilde{C}_{nm} \;\; = \;\; \int d^4x \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]_{ij} \varphi_{m,j}(x).$$

In a similar manner we can prove the identity

$$\int d^4x \; \varphi^\dagger_{n,i}(x) \left[i\alpha\gamma_5\right]^N_{ij} \varphi_{m,j}(x) = \tilde{C}^N_{nm}. \tag{16}$$

Therefore we can rewrite Eq. (14) as

$$\int d^4x\ \varphi^\dagger_{n,i}(x) \left(\mathbf{e}^{i\alpha\gamma_5}\right)_{ij} \varphi_{m,j}(x) \;=\; \delta_{nm} + \tilde{C}_{nm} + \frac{1}{2!}\tilde{C}^2_{nm} + \frac{1}{3!}\tilde{C}^3_{nm} + \cdots\ . \tag{17}$$

Then we use $\mathbf{det}A = \mathbf{exp}\{\mathbf{Tr}\ \mathbf{ln}A\}$ to obtain

$$\begin{aligned} J^{-1} &= \mathbf{det}\left\{\int d^4x\ \varphi^\dagger_{n,i}(x)\left(\mathbf{e}^{i\alpha\gamma_5}\right)_{ij}\varphi_{m,j}(x)\right\}^{-2} \\ &= \mathbf{exp}\left\{-2\sum_n^\infty \int d^4x\ \varphi^\dagger_{n,i}(x)\left[i\alpha\gamma_5\right]_{ij}\varphi_{n,j}(x)\right\}. \end{aligned} \tag{18}$$

Note that in obtaining Eq. (18) we did not assume that the transformation parameter α is infinitesimal.

Next we use

$$\lim_{\Lambda\to\infty} \mathbf{e}^{-(\lambda_n/\Lambda)^2} \tag{19}$$

as a regulator for the divergent summation, and we find

$$\begin{aligned} J^{-1} &= \mathbf{exp}\left\{-2\lim_{\Lambda\to\infty}\sum_n^\infty \int d^4x\ \varphi^\dagger_{n,i}(x)[i\alpha\gamma_5]_{ij}\mathbf{e}^{-(\lambda_n/\Lambda)^2}\varphi_{n,j}(x)\right\} \\ &= \mathbf{exp}\left\{-2i\lim_{\Lambda\to\infty}\mathbf{Tr}\int\frac{d^4k}{(2\pi)^4}\int d^4x\ \mathbf{e}^{-ikx}\alpha\gamma_5\mathbf{e}^{-(\not{D}/\Lambda)^2}\mathbf{e}^{ikx}\right\} \\ &= \mathbf{exp}\left\{-2i\lim_{\Lambda\to\infty}\mathbf{Tr}\int\frac{d^4k}{(2\pi)^4}\int d^4x\ \alpha\gamma_5\right. \\ &\qquad\qquad \left.\times\,\mathbf{exp}\left\{\frac{1}{\Lambda^2}\left[-(ik_\mu+D_\mu)^2 - \frac{i}{4}[\gamma^\mu\ \gamma^\nu]F_{\mu\nu}\right]\right\}\right\}. \end{aligned} \tag{20}$$

Since we are working in the Euclidean space time, we have the metric $g^{\mu\nu} = diag(-1,-1,-1,-1)$. As it is well known [18], the limiting procedure yields

$$J^{-1} = \mathbf{exp}\left\{-i\int d^4x\ \frac{\alpha}{16\pi^2}\mathbf{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\right\}. \tag{21}$$

In this way we can define anomaly of discrete symmetries.

3 Anomaly of Non-Abelian Discrete Family Symmetries

We start with the Lagrangian

$$\mathcal{L} = i\bar{\psi}\ \not{D}\psi - \frac{1}{2g_L^2}\mathbf{Tr}\ F^{\mu\nu}(L)F_{\mu\nu}(L) - \frac{1}{2g_R^2}\mathbf{Tr}\ F^{\mu\nu}(R)F_{\mu\nu}(R), \tag{22}$$

$$D_\mu = \partial_\mu - iL^a_\mu T^a_L P_L - iR^b_\mu T^b_R P_R, \tag{23}$$

which describes an $SU(N_L) \times SU(N_R)$ chiral Yang-Mills theory. Here $L^a(R^a)$ are gauge bosons that couple to the left- (right-) handed fermions, and P_L and P_R are the projection operators on the left- and right- handed pieces, respectively. Then we consider a non-abelian discrete chiral transformation

$$\psi_{i,\alpha'}(x) \to \psi'_{i,\alpha} = \left[\mathbf{e}^{iXP_L+iYP_R}\right]_{ij,\alpha\beta} \psi_{j,\beta}, \tag{24}$$

where X and Y are matrices that act on the family indices $\alpha, \beta = 1 \sim 3$. The transformation is a unitary transformation which does not mix the left and right-handed fields. Accordingly, we define the following phases:

$$\mathbf{e}^{i\alpha_L} \equiv \mathbf{det}(\mathbf{e}^{iX}), \qquad \mathbf{e}^{i\alpha_R} \equiv \mathbf{det}(\mathbf{e}^{iY}), \tag{25}$$

$$\alpha_L \equiv \mathbf{Tr}(X), \qquad \alpha_R \equiv \mathbf{Tr}(Y). \tag{26}$$

In contrast to the previous case, we introduce two complete sets of eigen states:

$$\not{D}^\dagger \not{D}\varphi_n = \lambda_n^2\varphi_n, \qquad \not{D}\not{D}^\dagger\phi_n = \lambda_n^2\phi_n, \tag{27}$$

$$\psi_{i,\alpha}(x) = \sum_n a_n\varphi_{n,i,\alpha}(x), \qquad \bar{\psi}_{i,\alpha}(x) = \sum_n \bar{b}_n\phi^\dagger_{n,i,\alpha}(x). \tag{28}$$

By using these expressions, we then calculate the Jacobian for the transformation (24), where we denote the Jacobian for $\mathcal{D}\psi$ by j and that for $\mathcal{D}\bar{\psi}$ by $\bar{j}$, so that the total Jacobian J is given by $\bar{j}j$. After similar calculations as in the abelian case, we find that

$$\begin{aligned} j^{-1} &= \left\{\mathbf{det}\int d^4x\ \varphi^\dagger_{n,i,\alpha}(x)\left[\mathbf{e}^{iXP_L+iYP_R}\right]_{ij,\alpha\beta}\varphi_{m,j,\beta}(x)\right\}^{-1} \\ &= \mathbf{exp}\left\{-i\int d^4x\ \varphi^\dagger_{n,i,\alpha}(x)\left[XP_L+YP_R\right]_{ij,\alpha\beta}\varphi_{m,j,\beta}(x)\right\} \\ &= \mathbf{exp}\left\{-i\lim_{\Lambda\to\infty}\mathbf{Tr}\int\frac{d^4k}{(2\pi)^4}\int d^4x\ \mathbf{e}^{-ikx}\left[XP_L+YP_R\right]\mathbf{e}^{-(\not{D}^\dagger\not{D}/\Lambda^2)}\mathbf{e}^{ikx}\right\}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \bar{j}^{-1} &= \left\{\mathbf{det}\int d^4x\ \phi^\dagger_{n,i,\alpha}(x)\left[\mathbf{e}^{-iXP_R-iYP_L}\right]_{ij,\alpha\beta}\phi_{m,j,\beta}(x)\right\}^{-1} \\ &= \mathbf{exp}\left\{i\int d^4x\ \phi^\dagger_{n,i,\alpha}(x)\left[XP_R+YP_L\right]_{ij,\alpha\beta}\phi_{m,j,\beta}(x)\right\} \\ &= \mathbf{exp}\left\{i\lim_{\Lambda\to\infty}\mathbf{Tr}\int\frac{d^4k}{(2\pi)^4}\int d^4x\ \mathbf{e}^{-ikx}\left[XP_R+YP_L\right]\mathbf{e}^{-(\not{D}\not{D}^\dagger/\Lambda^2)}\mathbf{e}^{ikx}\right\}, \end{aligned} \tag{30}$$

where $\mathbf{Tr}$ stands for the trace in the spinor, family and Yang-Mills indices. The trace in the family space can be carried out to obtain

$$j^{-1} = \mathbf{exp}\left\{-i\lim_{\Lambda\to\infty}\mathbf{Tr}\int\frac{d^4k}{(2\pi)^4}\int d^4x\ \mathbf{e}^{-ikx}\left[\alpha_LP_L+\alpha_RP_R\right]\mathbf{e}^{-(\not{D}^\dagger\not{D}/\Lambda^2)}\mathbf{e}^{ikx}\right\}, \tag{31}$$

where use have been made of Eqs. (25) and (26). In the same way we obtain a similar expression for $\bar{j}^{-1}$. The rest of the calculations are very similar to those of the previous case, and we finally obtain the total Jacobian

$$J^{-1} = j^{-1}\bar{j}^{-1} = \mathbf{exp}\left\{i\int d^4x\ \frac{1}{32\pi^2}\mathbf{Tr}\ \epsilon^{\mu\nu\rho\sigma}\left[\alpha_L\ F_{\mu\nu}(L)F_{\rho\sigma}(L) - \alpha_R\ F_{\mu\nu}(R)F_{\rho\sigma}(R)\right]\right\}, \tag{32}$$

where

$$F_{\mu\nu}(L) = \partial_\mu L_\nu - \partial_\nu L_\mu - i[L_\mu \ L_\nu], \tag{33}$$

$$F_{\mu\nu}(R) = \partial_\mu R_\nu - \partial_\nu R_\mu - i[R_\mu \ R_\nu]. \tag{34}$$

$\alpha_{L,R}$ are the phases of the transformation matrices defined in Eq. (25), and are not restricted to be continuous and infinitesimal. These phases correspond to the abelian parts of the non-abelian discrete family transformations. Therefore, to calculate anomaly of a non-abelian discrete family symmetry, we only have to take into account its abelian parts. We will discuss concrete examples in the later sections.

4 Pontryagin Index

The expression

$$\int d^4x \ \frac{1}{32\pi^2} \mathbf{Tr} \left[\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right] \tag{35}$$

in Eq. (32) is called the Pontryagin index, which is an integer ν. For the case $\alpha_L \neq 0$ and $\alpha_R = 0$, for instance, the Jacobian becomes

$$J^{-1} = \mathbf{exp}\{i\alpha_L \nu\}. \tag{36}$$

Since only the abelian parts of a non-abelian discrete family symmetry contribute to the anomalous Jacobian, the phase has the form

$$\alpha_L = \sum_I \ (2\pi/N_I) \sum_i q_{I,i}, \tag{37}$$

where we have assumed that the abelian parts are written as $\Pi_{I=1} Z_{N_I}$, and $q_{I,i}$ stands for the charge of Z_{N_I}. Therefore, we obtain $J^{-1} = 1$, if

$$\sum_i q_{iI,} = r N_I \quad (r = 0, \pm 1, \pm 2 \cdots) \tag{38}$$

is satisfied for each I.

5 Anomaly Cancellation and Gauge Coupling Unification

In the previous sections we have seen that anomaly of discrete symmetries can be defined as the anomalous Jacobian of the path-integral measure. In this section we want to discuss how the anomaly can be canceled by the GS(Green-Schwarz) mechanism [9, 10, 11]. First, we would like to recall the GS mechanism for an anomalous $U(1)_A$ symmetry, and then apply the GS mechanism to discrete symmetries.

5.1 Green-Schwarz Mechanism

String theory when compactified to four dimensions usually contains anomalous U(1) local symmetries. Consider a supersymmetric Yang-Mills theory based on a gauge group $\mathcal{G} \otimes U(1)_A$, where $U(1)_A$ is assumed to be anomalous. The $U(1)_A$ gauge transformation is defined as

$$\Phi \to \mathbf{e}^{-i\Lambda}\Phi, \tag{39}$$

$$V_A \to V_A + i(\Lambda - \bar{\Lambda}), \tag{40}$$

where Φ and Λ are chiral supermultiplets and V_A is the vector supermultiplet of $U(1)_A$. Anomaly for this transformation has been calculated in Ref. [19]. Using the result of [19], we find that the anomalous Jacobian for the $[\mathcal{G}]^2 \times U(1)_A$ anomaly, for instance, is given by

$$J^{-1} = \exp\left\{ -i \int d^4x d^2\theta \mathcal{A} \mathbf{Tr} \left[\Lambda W^a W_a \right]_F \right\}, \tag{41}$$

where $\mathcal{A}$ is the anomaly coefficient for $[\mathcal{G}]^2 \times U(1)_A$ and W is a chiral supermultiplet of the gauge supermultiplet corresponding to the gauge group $\mathcal{G}$. This anomaly can be canceled by the gauge kinetic term

$$k \mathbf{Tr} \left[S W^a W_a \right]_F, \tag{42}$$

if we correspondingly shift the dilaton supermultiplet S as

$$S \to S^{'} = S + i\frac{\mathcal{A}}{k}\Lambda, \tag{43}$$

where k is the Kac-Moody level. At the same time we have to modify the transformation property of the dilaton supermultiplet S to restore the invariance of its Kähler potential. At the quantum level, the Kähler potential is modified to include $\delta_{GS} V_A$ in the logarithm:

$$K = \mathbf{ln}(S + \bar{S} - \delta_{GS} V_A). \tag{44}$$

So, the Kähler potential is invariant, if the relation

$$\frac{\mathcal{A}}{k} = \delta_{GS} \tag{45}$$

is satisfied.

In the case that $\mathcal{G} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, there exist various possibilities of anomaly: $[SU(3)_C]^2 \times U(1)_A$, $[SU(2)_L]^2 \times U(1)_A$, $[U(1)_Y]^2 \times U(1)_A$, $[U(1)_A]^3$ and $[gravity]^2 \times U(1)_A$. If we denote the respective anomaly coefficients by $\mathcal{A}_3$, $\mathcal{A}_2$, $\mathcal{A}_1$, $\mathcal{A}_A$, and $\mathcal{A}_G$, then the anomaly cancellation conditions are given by

$$\frac{\mathcal{A}_3}{k_3} = \frac{\mathcal{A}_2}{k_2} = \frac{\mathcal{A}_1}{k_1} = \frac{\mathcal{A}_A}{k_A} = \frac{\mathcal{A}_G}{12} = \delta_{GS}. \tag{46}$$

Note that the Kac-Moody levels of a non-abelian group are positive integers, while there is no restriction on the Kac-Moody levels for an abelian group.

5.2 Gauge Coupling Unification

Next we look at the discussion above in terms of the component fields to find the relation of the Kac-Moody levels with the gauge coupling unification. To this end, we define $S|_{\theta=\bar{\theta}=0} = \varphi + i\eta$ and $\Lambda|_{\theta=\bar{\theta}=0} = \phi + i\xi$. On one hand, the shift (43) corresponds for the axion field η to

$$\eta \rightarrow \eta^{'} = \eta + \phi\delta_{GS}, \tag{47}$$

and the VEV of the dilaton field φ is nothing but the string coupling, i.e.,

$$< \varphi >= \frac{1}{g_{st}^2}. \tag{48}$$

On the other hand, the gauge couplings g_i are related to the string coupling according to

$$\frac{k_i}{g_{st}^2} = \frac{1}{g_i^2}. \tag{49}$$

Therefore, the conditions of the gauge coupling unification for the SM gauge couplings can be written as

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{st}^2, \tag{50}$$

at the string scale. It is therefore clear that the anomaly cancellation conditions (46) have a non-trivial influence on the gauge coupling unification.

5.3 The GS mechanism for Discrete Symmetries

Here we would like to extend the GS mechanism to the case of discrete symmetries. Unlike to [14], we do not assume that the discrete symmetry in question arises from a spontaneous break down of a continuous local symmetry. We instead assume that an anomalous discrete symmetry at low energy is remnant of an anomaly free discrete symmetry, and that its low energy anomaly is cancelled by the GS mechanism at a more fundamental scale. In superstring theory when compctified on a six dimensional Calabi-Yau manifold, for instance, there exist indeed certain non-abelian discrete symmetries [21].

Consider a Z_N transformation

$$\begin{aligned} &\Phi \rightarrow \mathbf{e}^{-i\alpha}\Phi, \\ &V \rightarrow V, \end{aligned} \tag{51}$$

in a supersymmetric gauge theory to find out how to cancel anomaly. As before, the transformation parameter α is a discrete parameter, i.e. $\alpha = \frac{2\pi}{N}$, and V is the vector supermultiplet of the gauge group. Anomaly of this transformation has the same form as (41):

$$J^{-1} = \exp\left\{-i\int d^4xd^2\theta\mathcal{A}\mathbf{Tr}\left[\alpha W^a W_a\right]_F\right\}, \tag{52}$$

which can also be canceled by the gauge kinetic term. In this case, however, the dilaton supermultiplet has to be shifted by only a constant amount α, i.e.,

$$S \to S^{'} = S + i\frac{\mathcal{A}}{k}\alpha, \tag{53}$$

for the cancellation mechanism to work. This means that because α is a constant independent of x and θ, only the imaginary part of scalar component of S, which is the axion field, should be shifted. Note that the Kähler potential (44) is invariant whatever δ_{GS} is, because the vector supermultiplet does not change under the transformation Eq. (51). Therefore, the anomaly cancellation conditions for the SM gauge group are:

$$\frac{\mathcal{A}_3}{k_3} = \frac{\mathcal{A}_2}{k_2} = \frac{\mathcal{A}_1}{k_1} = \frac{\mathcal{A}_G}{12}, \tag{54}$$

where $\mathcal{A}_3$, $\mathcal{A}_2$, $\mathcal{A}_1$ and $\mathcal{A}_G$ are the anomaly coefficients of the anomalies $[SU(3)_C]^2 \times Z_N$, $[SU(2)_L]^2 \times Z_N$, $[U(1)_Y]^2 \times Z_N$ and $[gravity]^2 \times Z_N$, respectively. Since the $[U(1)_Y]^2 \times Z_N$ anomaly does not yield useful constraints on the low-energy effective theory, and we cannot calculate $\mathcal{A}_G$ for the low-energy effective theory, we will not consider them when discussing models in the next section. However, massive Majorana fields can contribute to $\mathcal{A}_2$ and $\mathcal{A}_3$ for even N, because Majorana masses can be allowed by the discrete symmetry if the Majorana fields belong to real representations of $SU(2)_L$ and $SU(3)_C$ [14]. Taking into account the contributions from the massive fields, we arrive at the anomaly cancellation conditions [14]

$$\frac{\mathcal{A}_3 + \frac{pN}{2}}{k_3} = \frac{\mathcal{A}_2 + \frac{qN}{2}}{k_2} \tag{55}$$

with integer p and q, where $pN/2$ and $qN/2$ take into account the possible contributions from the heavy fields.

In this section, we have applied the GS mechanism to an abelian discrete symmetry. In sect. 7, we will consider anomaly cancellation of a non-abelian discrete family symmetry. As we have seen in sect. 3, however, only the abelian parts contribute to anomaly even if we consider a non-abelian discrete family symmetry . Hence, Eq. (55) can be applied to the case of a non-abelian discrete family symmetries, too.

6 Anomaly of Accidental Z_N Symmetries

As an example of anomaly cancellation of discrete symmetries, let us consider the Baryon and Lepton number symmetries in the MSSM with R-parity, in which see-saw mechanism is implemented to generate neutrino masses. The Baryon number $U(1)_B$ is conserved at the classical level, while the Lepton number $U(1)_L$ is violated because of Majorana masses of the right-handed neutrinos. However, its abelian discrete subgroups $(Z_N)_L$ with even N are still intact at the classical level. In the following discussion we first investigate anomalies and its GS cancellation conditions for abelian discrete subgroups $(Z_M)_B$ of $U(1)_B$ and $(Z_N)_L$. Then we see how the GS cancellation conditions influence

	Q	U^c	D^c	L	E^c	ν_R	H^u	H^d
$(Z_N)_L$	aN	bN	cN	$\frac{N}{2}+dN$	$\frac{N}{2}+eN$	$\frac{N}{2}+fN$	gN	hN
$(Z_M)_B$	$B+iM$	$-B+jM$	$-B+kM$	lM	mM	nM	oM	pM

Table 1: The $(Z_N)_L$ and $(Z_N)_B$ charges. N is even and $a \sim p = 0, 1, 2 \cdots$.

gauge coupling unification. The $(Z_N)_L$ and $(Z_M)_B$ charges of the supermultiplets of the MSSM are given in Table 1.

The anomaly coefficients for $(Z_N)_L$ are calculated to be

$$2\mathcal{A}_3 = \frac{N}{2}[12a + 6b + 6c], \tag{56}$$

$$2\mathcal{A}_2 = \frac{N}{2}[18a + 3 + 6d + 2g + 2h]. \tag{57}$$

Therefore, the GS cancellation conditions become

$$\frac{k_3}{k_2} = \frac{12a + 6b + 6c}{18a + 3 + 6d + 2g + 2h} = \frac{even}{odd}. \tag{58}$$

Note that $k_3 = k_2 = 1$ is NOT a solution. In a similar way, we can calculate anomaly coefficients for $(Z_M)_B$ and find

$$\frac{k_3}{k_2} = \frac{9B + (9i + 3\ell + o + p)M}{(6i + 3j + 3k)M} = \frac{even\ \textbf{or}\ odd}{even\ \textbf{or}\ odd}. \tag{59}$$

Since it is believed to be difficult to build realistic models with higher Kac-Moody levels in string theory, we look for solutions to (58) and (59) with lower levels. The solution with the lowest levels that can satisfy the conditions (58) and (59) simultaneously is $k_3 = 2, k_2 = 1$, yielding gauge coupling unification condition

$$2g_3^2 = g_2^2 = k_1 g_1^2 = g_{st}^2, \tag{60}$$

where k_1 is arbitrary. Fig. 1 shows the ratio of g_2^2/g_1^2 (upper line) and g_2^2/g_3^2 (lower line) as a function of energy scale. The solid lines correspond to the case with one pair of $SU(2)_L$ doublet Higgs supermultiplets, the dotted lines to the case with two pairs and the dashed lines to the case with three pairs. (We denote the number of the Higgs pairs by $H_{\rm higgs}$.) We see from Fig. 1 that the ratio g_2^2/g_3^2 with $H_{\rm higgs} = 3$ becomes close to 2 at the Planck scale $M_{PL} = 1.2 \times 10^{18}$ GeV. For $H_{\rm higgs} = 1$ and 2 there is no chance for the ratio to become close to 2 below or near M_{PL}. In Fig. 2 we plot the running of $(\alpha_1 k_1)^{-1}, (\alpha_2 k_2)^{-1}$ and $(\alpha_3 k_3)^{-1}$ with $k_3 = 2, k_2 = 1, k_1 = 2.25$ in the case of $H_{\rm higgs} = 3$. As we have seen above, the GS cancellation conditions of anomalies have nontrivial influence on gauge coupling unification and hence the number of Higgs supermultiplets.

7 Models

Recently, a number of models with a non-abelian discrete family symmetry are proposed [1, 2, 3, 4, 5, 6, 7]. However, if only the SM Higgs or the MSSM Higgs are present within

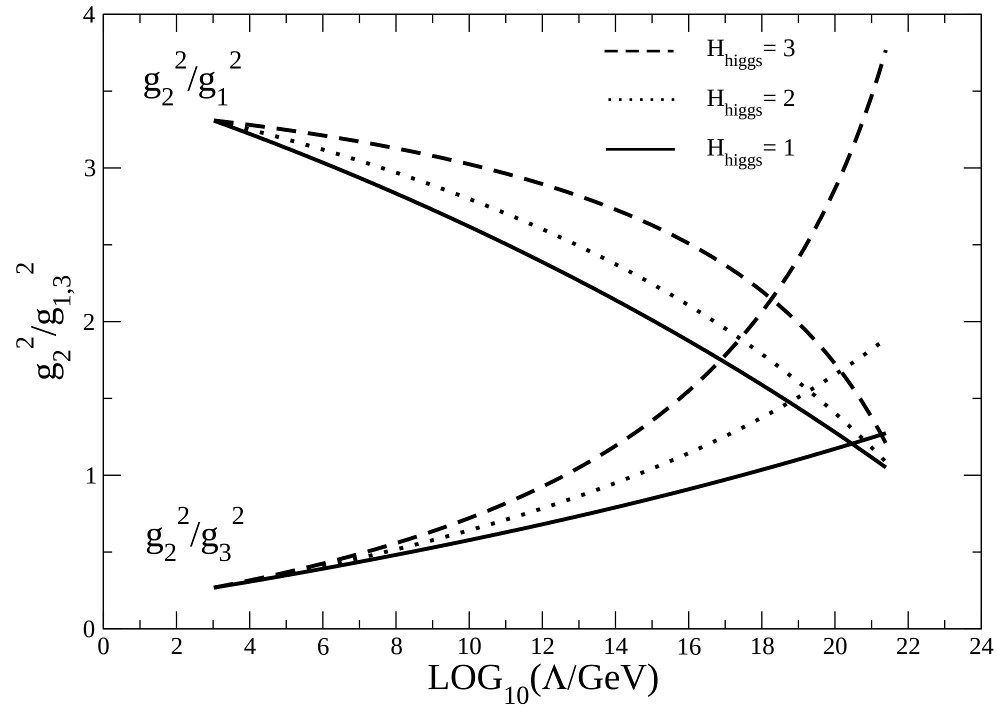

Figure 1: The ratio of g_2^2/g_1^2 (upper line) and g_2^2/g_3^2 (lower line) as a function of energy scale. The solid lines correspond to the MSSM+ν_R case, the dotted lines to the case with $H_{\rm higgs} = 2$ and the dashed lines to the case with $H_{\rm higgs} = 3$.

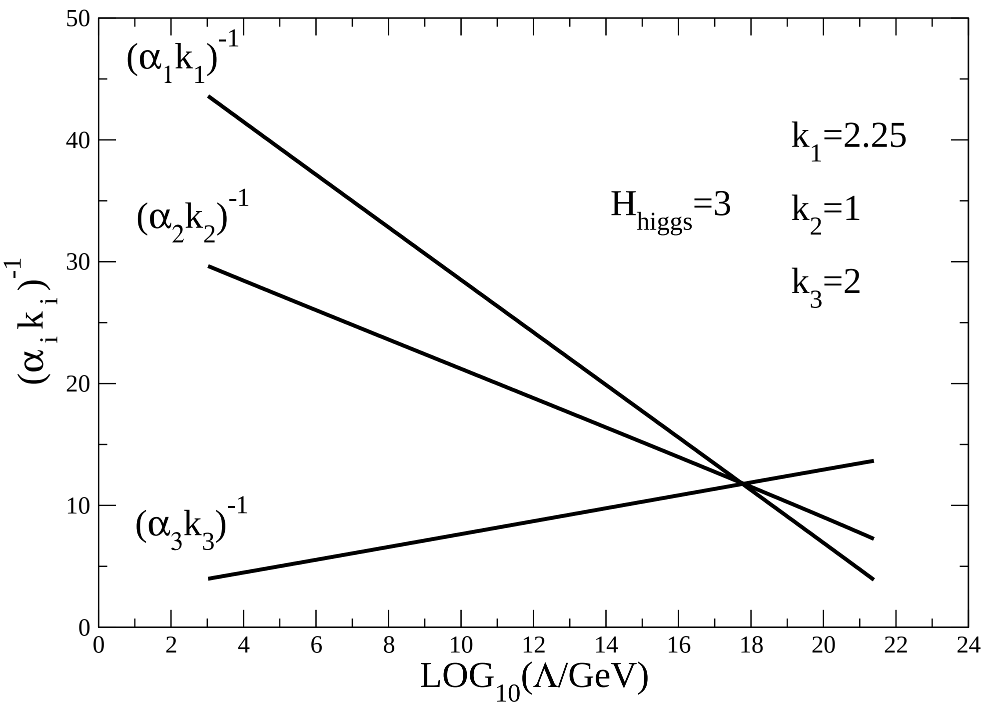

Figure 2: The running of $(\alpha_1 k_1)^{-1}$, $(\alpha_2 k_2)^{-1}$ and $(\alpha_3 k_3)^{-1}$ with $k_3 = 2, k_2 = 1$ and $k_1 = 2.25$ in the case with $H_{\rm higgs} = 3$. The unification scale is 10^{18} GeV.

the framework of renormalizable models, any low-energy non-abelian family symmetry should be hardly broken to be consistent with experimental observations. That is, if a non-abelian discrete family symmetry should be at most softly broken, we need several pairs of $SU(2)_L$ doublet Higgs fields. This implies that the conditions of the ordinary unification of gauge couplings, i.e. $k_2 = k_3 = k_1(3/5)$, cannot be satisfied if we insist on a low-energy non-abelian discrete family symmetry. Fortunately, as we have seen, there is a possibility to satisfy the gauge coupling unification conditions with a non- minimal Higgs content if the Kac-Moody levels assume non-trivial values. On the other hand, these Kac-Moody levels play an important role in the anomaly cancellation (GS mechanism), too. In the following subsections, we calculate anomalies of non-abelian discrete family symmetries for recent models and investigate the gauge coupling unification conditions.

7.1 D_7 Model

Let us first calculate anomaly of the supersymmetric D_7 model [5, 6]. D_7 has fourteen elements and five irreducible representations $(\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{2}', \mathbf{2}'')$. This model uses the complex representation [20], so the character table and the two-dimensional representation matrices of $\mathbf{2}$ are written as follows.

class	n	h	χ_1	$\chi_{1'}$	χ_2	$\chi_{2'}$	$\chi_{2''}$
C_1	1	1	1	1	2	2	2
C_2	7	2	1	-1	0	0	0
C_3	2	7	1	1	a_1	a_2	a_3
C_4	2	7	1	1	a_2	a_3	a_1
C_5	2	7	1	1	a_3	a_1	a_2

$a_k = 2cos(\frac{2\pi}{7}k)$

Table 2: Character table of D_7

$$
\begin{aligned}
C_1 &: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
C_2 &: \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix} \quad k = 0 \sim 6 \quad \omega = \exp(2\pi i/7) \\
C_3 &: \begin{pmatrix} \omega^6 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix} \\
C_4 &: \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix} \\
C_3 &: \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}
\end{aligned} \tag{61}
$$

The representation matrices for $\mathbf{2}'$ and $\mathbf{2}''$ are obtained by the cyclic rotation of $C_{3,4,5}$. D_7 has five kinds of transformation properties corresponding to five classes. However, the transformation of C_1 is the identity transformation, and there is no difference among $C_{3,4,5}$ when we calculate anomaly. Hence we consider only C_2 and C_3. Under C_2 and C_3, the irreducible representations transform as

$$
\begin{array}{llll}
 & C_2 & & C_3 \\
\mathbf{2} & \rightarrow \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix} \mathbf{2} & & \rightarrow \begin{pmatrix} \omega^6 & 0 \\ 0 & \omega^1 \end{pmatrix}, \begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^6 \end{pmatrix} \mathbf{2} \\
\mathbf{2}' & \rightarrow \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix} \mathbf{2}' & & \rightarrow \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix} \mathbf{2}' \\
\mathbf{2}'' & \rightarrow \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix} \mathbf{2}'' & & \rightarrow \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix} \mathbf{2}'' \\
\mathbf{1} & \rightarrow \mathbf{1} & & \rightarrow \mathbf{1} \\
\mathbf{1}' & \rightarrow \mathbf{e}^{i\frac{2\pi}{2}1}\ \mathbf{1}' & & \rightarrow \mathbf{1}',
\end{array}
\tag{62}
$$

where $k = 1 \sim 7, \omega = \exp(2\pi i/7)$. As we have mentioned in sect. 3, even if we calculate anomaly of a non-abelian discrete family symmetry, only the abelian parts contribute to anomaly. From Eqs. (61) and (62), it is clear that the abelian parts of C_2 and C_3 transformations are Z_2 and Z_7 respectively.

In this model, the authors of Ref. [5, 6] introduce the $SU(2)_L$ triplet extra Higgs supermultiplets, ξ_i, that have Lepton number, and assume that all Higgs supermultiplets have generation as well as fermions. The assignment of the D_7 representations for each matter supermultiplets is given by the following Table 3. [2]

	$Q_{1,2}\ D^c_{1,2}$	$U^c_{1,2}$	$L_{2,3}\ E^c_{2,3}$	$Q_3\ D^c_3\ U^c_3\ L_1\ E^c_1$	$H^d_{1,2}$	$H^u_{1,2}$	$\xi_{2,3}$	$H^d_3\ H^u_3\ \xi_1$
D_7	2	$2'$	$2''$	1	2	$2''$	$2''$	1

Table 3: D_7 assignment of the matter supermultiplets

7.1.1 Computation of anomaly coefficients

Let us compute the anomaly coefficients for $C_3(Z_7)$ transformations. These transformations have no anomaly because the determinants of all C_3 transformation matrices of Eq. (62) are equal to one.

Next let us compute the anomaly coefficients for $C_2(Z_2)$ transformations. The first and second generations of QD^c,U^c and H^uH^d are assigned to $\mathbf{2}, \mathbf{2}'$ and $\mathbf{2}''$ respectively, and the third generation is assigned to $\mathbf{1}$. Hence, these supermutiplets transform as

$$
\Psi_{\alpha=1\sim3} \rightarrow \begin{pmatrix} 0 & \omega^k & 0 \\ \omega^{7-k} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} \Psi_{\beta=1\sim3}. \tag{63}
$$

The determinant of this matrix is equal to $\exp(2\pi i/2) = -1$, therefore these supermultiplets contribute $\frac{2\pi}{2}1$ anomaly coefficients. Here we omit the Z_N factor, that is $\frac{2\pi}{N}$, so the

[2] In Ref. [5], assignment for leptons are not specified. So we assume the assignment for those supermultiplets given in Ref. [6].

contributions of these supermultiplets are equal to one. On the other hand, the second and third generations of L, E^c and ξ are assigned to $\mathbf{2}''$, and the first generation is assigned to $\mathbf{1}$. Hence, these supermultiplets transform as

$$\Phi_{\alpha=1\sim 3} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^k \\ 0 & \omega^{7-k} & 0 \end{pmatrix}_{\alpha\beta} \Phi_{\beta=1\sim 3}, \tag{64}$$

and contribute one as well. By using these facts, we can compute the anomaly coefficients and find

$$2\mathcal{A}_3 = [1\cdot 2+1+1] = 4\ (mod\ 2) \tag{65}$$
$$2\mathcal{A}_2 = [1\cdot 3+1+1+1] + 1\cdot 4\cdot 2 = 14\ (mod\ 2). \tag{66}$$

Here, we define $\mathcal{A}_3$ and $\mathcal{A}_2$ as the anomaly coefficients of $[SU(3)_C]^2\times Z_2$ and $[SU(2)_L]^2\times Z_2$ respectively. The last term in Eq. (66) is the contribution from the $SU(2)_L$ triplet Higgs supermultiplets. As we have seen in sect. 4, these coefficients do not contribute anomaly. Therefore this model is anomaly free.

7.2 A_4 Model

As the second example, let us calculate anomaly of the supersymmetric A_4 model [1]. A_4 has twelve elements and four irreducible representations $(\mathbf{1},\mathbf{1}',\mathbf{1}'',\mathbf{3})$. This model uses the complex representation too, so the character table and the tree-dimensional representation matrices of $\mathbf{3}$ are written as follows.

class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

$\omega = \mathbf{e}^{i\frac{2\pi}{3}}$

Table 4: Character table of A_4

$$\begin{aligned}
C_1 &: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
C_2 &: \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
C_3 &: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \\
C_4 &: \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{67}$$

A_4 has four kinds of transformation properties corresponding to four classes. However, there is no difference between C_2 and C_3 when we calculate anomaly. The abelian parts of $C_{2,3}$ and C_4 are Z_3 and Z_2 respectively.

The assignment of the A_4 representations for the matter supermultiplets are given in Table 5. In addition, the authors of Ref. [1] introduce extra Leptons, Quarks and Higgs supermultiplets given in Table 6.

	$Q_{1,2,3}$ $L_{1,2,3}$	U_1^c D_1^c E_1^c	U_2^c D_2^c E_2^c	U_3^c D_3^c E_3^c	H^u H^d
A_4	3	1	$1^{'}$	$1^{''}$	1

Table 5: A_4 assignment of the matter supermultiplets

	$u_{1,2,3}$ $u^c_{1,2,3}$	$d_{1,2,3}$ $d^c_{1,2,3}$	$e_{1,2,3}$ $e^c_{1,2,3}$	$N^c_{1,2,3}$	$\chi_{1,2,3}$
A_4	3	3	3	3	3
$SU(3)_C$	3	3	1	1	1

Table 6: A_4 and $SU(3)_C$ assignment of extra supermultiplets

7.2.1 Computation of anomaly coefficients

Let us compute anomaly coefficients in the same way as in the case of D_7. For $C_{3(2)}$ transformation, the supermultiplets which are assigned to **3** and **1** do not contribute to anomaly because the determinant of these transformation matrices are equal to 1. Therefore, only $\mathbf{1}^{'}$ and $\mathbf{1}^{''}$ contribute to the anomaly coefficients. As we can see from Table 5, the three generations of the right-handed Quark and Lepton supremultiplets are assigned to $\mathbf{1}, \mathbf{1}''$ and $\mathbf{1}^{'''}$, respectively and so transform as

$$1st \quad \ddagger \quad \mathbf{1} \to \mathbf{1} \tag{68}$$

$$2nd \quad \ddagger \quad \mathbf{1}^{'} \to \mathbf{e}^{i\frac{2\pi}{3}2}\mathbf{1}^{'} \tag{69}$$

$$3rd \quad \ddagger \quad \mathbf{1}^{''} \to \mathbf{e}^{i\frac{2\pi}{3}1}\mathbf{1}^{''}. \tag{70}$$

Therefore these representations do not contribute to anomaly when we take into account all generations.

On the other hand, there is no anomaly for C_4 transformation because all singlets do not transform and the determinant of all transformation matrices of **3** are equal to 1. Therefore this model is anomaly free.

7.3 Q_6 Model

Let us finally calculate anomaly of the supersymmetric Q_6 model [4]. All elements of Q_6 are made from combinations of

$$A_{Q_6} = \begin{pmatrix} cos\ \phi_6 & sin\ \phi_6 \\ -\sin\ \phi_6 & cos\ \phi_6 \end{pmatrix}_{\phi_6 = \frac{2\pi}{6}} \qquad B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{71}$$

as follows.

$$\mathcal{G} = \{E, A_{Q_6}, (A_{Q_6})^2, \cdots, (A_{Q_6})^5, B_Q, A_{Q_6}B_Q, (A_{Q_6})^2 B_Q, \cdots, (A_{Q_6})^5 B_Q\} \tag{72}$$

E is the identity element. Q_6 has six irreducible representations, two doublets and four singlets:

$$\textit{doublet} \;\ddagger\; \mathbf{2} \quad \mathbf{2}' \tag{73}$$
$$\textit{singlet} \;\ddagger\; \mathbf{1} \quad \mathbf{1}' \quad \mathbf{1}'' \quad \mathbf{1}'''. \tag{74}$$

The transformation properties of each irreducible representations are characterized by A_{Q_6} and B_Q. For example, it can be written as follows.

$$\begin{array}{lll}
 & A_{Q_6} & B_Q \\
\mathbf{2} & \to \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \mathbf{2} & \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathbf{2} \\
\mathbf{2}' & \to \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \mathbf{2}' & \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{2}' \\
\mathbf{1} & \to \mathbf{1} & \to \mathbf{1} \\
\mathbf{1}' & \to \mathbf{1}' & \to \mathbf{e}^{i\frac{2\pi}{4}2}\ \mathbf{1} \\
\mathbf{1}'' & \to \mathbf{e}^{i\frac{2\pi}{6}3}\ \mathbf{1}'' & \to \mathbf{e}^{i\frac{2\pi}{4}3}\ \mathbf{1}'' \\
\mathbf{1}''' & \to \mathbf{e}^{i\frac{2\pi}{6}3}\ \mathbf{1}''' & \to \mathbf{e}^{i\frac{2\pi}{4}1}\ \mathbf{1}'''
\end{array} \tag{75}$$

It is clear that the abelian parts of A_{Q_6} and B_Q transformations are equal to Z_6 and Z_4, respectively because

$$(A_{Q_6})^6 = E \qquad (B_Q)^4 = E. \tag{76}$$

The assignment of Q_6 representations of the matter supermultiplets are given in Table 7.

	$Q_{1,2}\ L_{1,2}$	$U^c_{1\ 2}\ D^c_{1,2}\ E^c_{1,2}\ N^c_{1,2}$	$H^u_{1,2}\ H^d_{1,2}$	$Q_3\ L_3$	$U^c_3\ D^c_3\ E^c_3\ N^c_3$	$H^u_3\ H^d_3$
Q_6	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{2}'$	$\mathbf{1}'$	$\mathbf{1}'''$	$\mathbf{1}'''$

Table 7: Q_6 assignment of the matter supermultiplets

7.3.1 Computation of anomaly coefficients

$$\Psi_{\alpha=1\sim3} \to \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} \Psi_{\beta=1\sim3} \tag{77}$$

is the A_{Q_6} transformation property of Q and L. corresponding to A_{Q_6}. Because the determinant of this matrix is equal to one, Q and L do not contribute to anomaly for this transformation. On the other hand, U^c, D^c, E^c, N^c, H^u and H^d transform as

$$\Psi_{\alpha=1\sim3} \rightarrow \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & \mathbf{e}^{i\frac{2\pi}{6}3} \end{pmatrix}_{\alpha\beta} \Psi_{\beta=1\sim3}. \tag{78}$$

These supermultiplets contribute three to anomaly coefficients, and the anomaly coefficients are found to be

$$2\mathcal{A}_3 = 0\cdot 2 + 3 + 3 = 6\ (mod\ 6) \tag{79}$$
$$2\mathcal{A}_2 = 0\cdot 3 + 0 + 3 + 3 = 6\ (mod\ 6). \tag{80}$$

As we have discussed in sect. 4, these coefficients do not contribute to anomaly.

In the same way, we can compute anomaly for the transformation corresponding to B_Q. Under this transformation, Q and L transform as

$$\Psi_{\alpha=1\sim3} \rightarrow \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & \mathbf{e}^{i\frac{2\pi}{4}2} \end{pmatrix}_{\alpha\beta} \Psi_{\beta}, \tag{81}$$

and U^c, D^c, E^c, N^c, H^u and H^d transform as

$$\Psi_{\alpha=1\sim3} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathbf{e}^{i\frac{2\pi}{4}1} \end{pmatrix}_{\alpha\beta} \Psi_{\beta}. \tag{82}$$

The anomaly coefficients can be computed to

$$2A_3 = 2\cdot 2 - 1 - 1 = 2\ (mod\ 4) \tag{83}$$
$$2A_2 = 2\cdot 3 + 2 - 1 - 1 = 6\ (mod\ 4). \tag{84}$$

Therefore the Z_4 part is anomalous.

7.3.2 Cancellation of Anomalies and Gauge Coupling Unification

As we have seen in above sub-sections, the Z_4 part is anomalous, while the Z_6 part is anomaly free. This anomaly can be canceled by the GS mechanism, if

$$\frac{1\ (mod\ 2)}{k_3} = \frac{3\ (mod\ 2)}{k_2} \tag{85}$$

is satisfied. For example, it is possible if the values of the Kac-Moody levels are $k_3 = k_2$ or $k_2 = 1(3), k_3 = 3(1)$. Fig. 3 and 1 show the ratio of gauge couplings. As for the cases of $k_3 = k_2$ and $k_2 = 3, k_3 = 1$, the unification point of $SU(3)_C$ and $SU(2)_L$ gauge couplings is far lower than the Planck scale. In the case of $k_2 = 1, k_3 = 3$, the $SU(3)_C$ and $SU(2)_L$ gauge coupling constants can be unified at a scale slightly higher than the Planck scale. In this case, it is also possible to unify $U(1)_Y$ at the same point, if we assume $k_1 \simeq 1.63$. Fig. 4 is the running of gauge couplings in the case of $k_3 = 3, k_2 = 1, k_1 \simeq 1.63$.

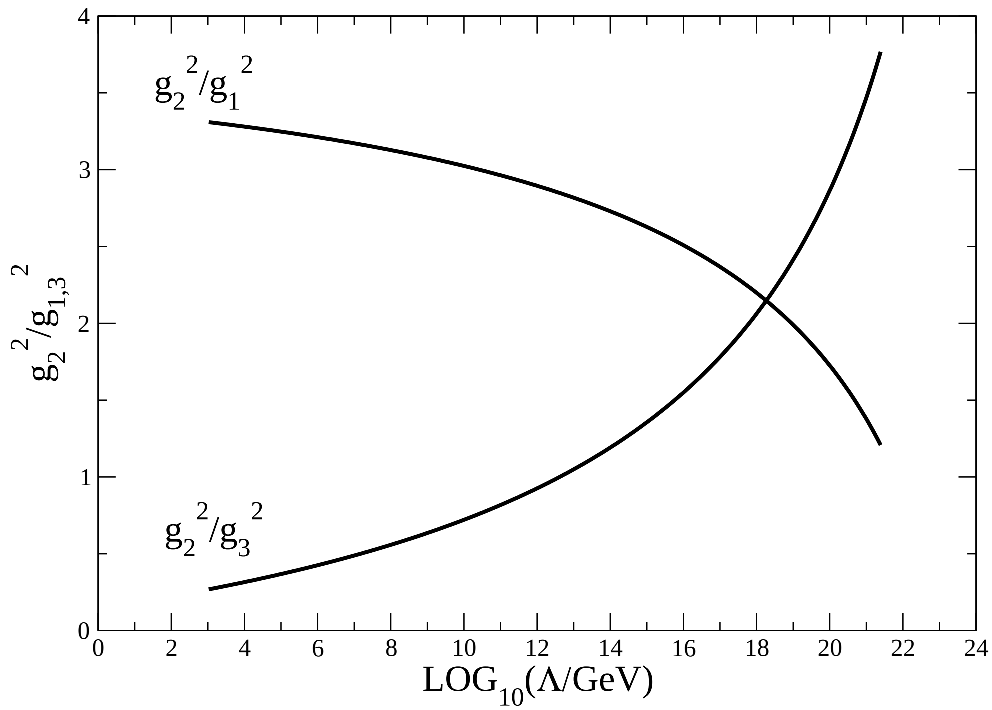

Figure 3: The ratio of g_2^2/g_1^2 (upper line) and g_2^2/g_3^2 (lower line) as a function of energy scale.

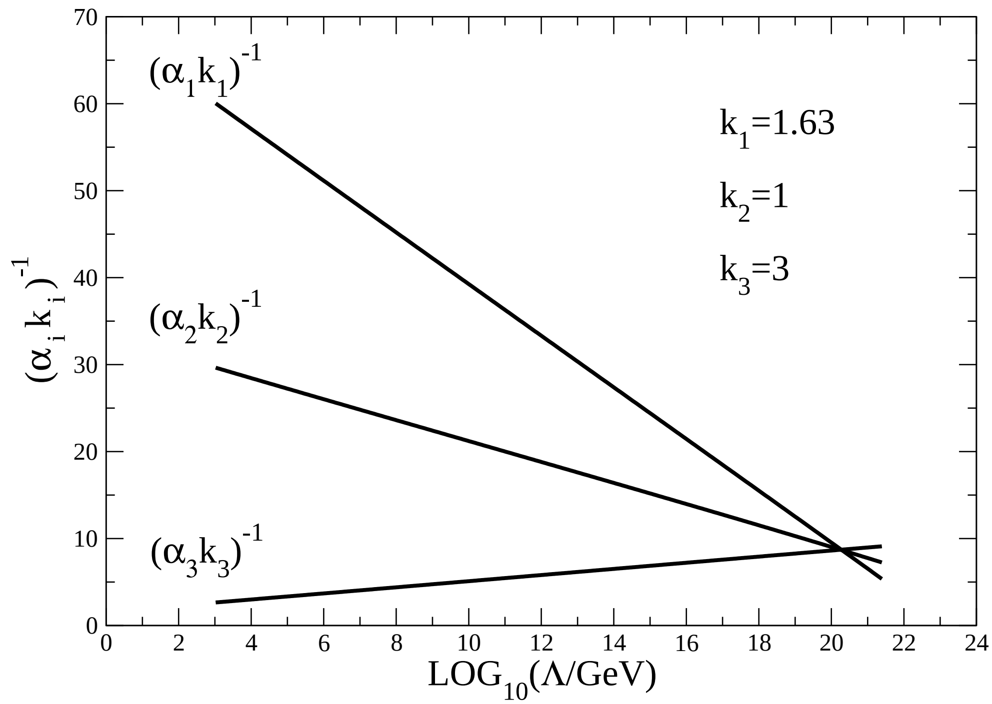

Figure 4: The running of $(\alpha_1 k_1)^{-1}$, $(\alpha_2 k_2)^{-1}$ and $(\alpha_3 k_3)^{-1}$ with $k_3 = 3, k_2 = 1$ and $k_1 \simeq 1.63$. The unification scale is 10^{20} GeV.

8 Summary

In this paper, we have investigated anomaly of discrete symmetries and their cancellation mechanism. We have seen that anomalies of discrete symmetries can be defined as the anomalous Jacobian of the path-integral measure, and that if we assume anomalies should be canceled by the GS mechanism, the ordinary conditions of gauge coupling unification can be changed. As for discrete abelian Baryon number and Lepton number symmetries in the MSSM with see-saw mechanism, we find that the ordinary unification conditions of gauge couplings are not consistent with the GS cancellation conditions, and that the existence of three pairs of $SU(2)_L$ Higgs doublets is a possible solution to satisfy the GS cancellation conditions and the unification conditions simultaneously.

We have investigated the cases of recently proposed supersymmetric models with a non-abelian discrete family symmetry. In the examples in this paper the gauge couplings do not exactly meet at the Planck scale, but we think that the examples are suggesting a right direction. If we take into account the threshold corrections at M_{PL}, for instance, the conditions could be exactly satisfied.

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